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Dirac particle in a box, and relativistic quantum Zeno dynamics

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Abstract

After developing a complete set of eigenfunctions for a Dirac particle restricted to a box, the quantum Zeno dynamics of a relativistic system is considered. The evolution of a continuously observed quantum mechanical system is governed by the theorem put forth by Misra and Sudarshan. One of the conditions for quantum Zeno dynamics to be manifest is that the Hamiltonian is semi-bounded. This Letter analyzes the effects of continuous observation of a particle whose time evolution is generated by the Dirac Hamiltonian. The theorem by Misra and Sudarshan is not applicable here since the Dirac operator is not semi-bounded.

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1. Introduction

The central purpose of this Letter is to point out the effects of continuous measurement of a relativistic quantum system. In [1], the authors have listed the necessary requirements of the Hamiltonian operator to exhibit Quantum Zeno Dynamics (QZD). One of the stipulation under which the theorem applies is that the Hamiltonian be lower semi-bounded. This criterion is readily met by a non-relativistic electron. The Schrödinger equation restricts the energy eigenvalues of a quantum system to be bounded from below. Thus, as is shown in [2], the non-relativistic electron does indeed exhibit QZD. In the body of our Letter we will make it clear as to what it means for a quantum mechanical system to exhibit QZD.

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The Dirac operator does permit negative energy solutions, and this makes the operator in general unbounded. It is essential to point out that, just because the non-relativistic electron does in fact demonstrate QZD, there is no reason to expect the particle described by the Dirac equation (as opposed to the second quantized Hamiltonian) to be subject to the same requirements. After all, the negative energy states have to be reinterpreted to formulate a consistent theory of relativistic electrons as in quantum electrodynamics. Nonetheless, it should not be shocking to realize that the Dirac operator does in fact permit QZD, as will be shown in the latter half of the Letter.

As Facchi et al. has shown (see [2]) for the non-relativistic case, considerable simplification in calculation occurs if we have with us a complete list of eigenfunctions for the "particle in a box" Hamiltonian. For the non-relativistic case, these eigenfunctions are well known (for example, see [3]). With this in mind, we construct a complete system of eigenfunctions for the Dirac particle in a box in the first part of the Letter. In [4,5], the authors do in fact compute the positive spectrum for the particle in a box. Our calculations require the complete spectrum (i.e., including the negative energy states). In addition, in order to construct a complete spectrum, we do not use the same boundary conditions as in [4,5]. In fact, boundary conditions are not used at all to obtain the eigenfunctions; instead, we subject the wavefunctions to "a priori" requirements, which will be justified at the end.

2. Dirac particle in a box

We seek to construct an explicit list of eigenstates for a relativistic particle confined to an infinitely deep square well. Our calculations will be restricted to one spatial dimension. The Hamiltonian governing the system will be the classical Dirac operator combined with a suitable potential operator that will confine the states to a well. In 1D, the Dirac operator will be given by

$$H_{\mathrm{D}} = \gamma^0\left[-\mathrm{i}\gamma^3\frac{\partial}{\partial z} + m\right] \tag{1}$$

where $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ and $\gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}$, where σ_3 is a (2×2) Pauli matrix, and I is the (2×2) identity matrix (the representation used here is as given in the Appendix A of [6]). It is well known that [4,5], one-way to confine the states to a certain region, is to make the parameter m appearing in (1) be a function of position. The resulting modification of the Dirac operator will denoted by the "particle in a box" Hamiltonian: H_{Box}. As usual, the time evolution of the physical states must satisfy

$$H_{\mathrm{Box}}\psi(t,x) = (H_{\mathrm{D}} + V_0)\psi = \mathrm{i}\frac{\partial\psi}{\partial t}, \tag{2}$$

where H_{D} is defined by (1) and

$$V_0 = \gamma^0 M\left[1 - \tilde{\chi}_A(x)\right]. \tag{3}$$

The constant M will ultimately go to infinity, and $\tilde{\chi}_A$ is the characteristic function:

$$\tilde{\chi}_A = \begin{cases} 1, & x \in A, \text{ where } A = [0, L], \\ 0, & \text{otherwise}. \end{cases} \tag{4}$$

Our "box" separates space into three different regions; regions I, II, and III are defined by the inequalities $z < 0$, $0 \leqslant z \leqslant L$, and $z > L$, respectively. From Eqs. (2)–(4), it is clear that in the three separate regions, the wave function will satisfy the Dirac equation (albeit with a non traditional mass). We will build our eigenfunctions by piecing together eigenfunctions of the free Dirac operator. In regions I and III, the spatial dependence of the wavefunction is given by $e^{-\mathrm{i}k'x}$ and $e^{\mathrm{i}k'x}$, respectively (where $k' = \sqrt{E^2 - (M+m)^2}$) since we want our wavefunctions to be bounded at infinity. Clearly, in the infinite M limit we find that the wavefunctions outside the box vanishes as required. The positive energy eigenstates in region II are then given by

$$\psi_{+,j} = A_j e^{\mathrm{i}kz} \begin{pmatrix} \chi_j \\ \frac{\sigma_3 k}{E+m}\chi_j \end{pmatrix} + B_j e^{-\mathrm{i}kz} \begin{pmatrix} \chi_j \\ \frac{-\sigma_3 k}{E+m}\chi_j \end{pmatrix}. \tag{5}$$

Here, "+" refers to the positive energy eigenstates, and $j = 1$ refers to the spin up state and $j = 2$ refers to the spin down state. As usual, the two component spinor χ_j is given by $\chi_1 = \binom{1}{0}$ and $\chi_2 = \binom{0}{1}$. The energy eigenvalue is related to the parameter k as follows: $E = \sqrt{k^2 + m^2}$, and (2) implies that $\psi_{+,j}(z,t) = \psi_{+,j}(z)e^{-\mathrm{i}Et}$. Similar superposition states can be constructed for the negative energy states.

We wish to construct a Hamiltonian operator H_{Box} that is self-adjoint. Since the final Hilbert space will be defined as the closed linear span of all the eigenfunctions of H_{Box}, the operator will naturally be densely defined. The Hermitian property of H_{Box} requires that

$$\langle \psi_{\mathrm{I}} | H_{\mathrm{Box}} \psi_{\mathrm{II}} \rangle = \langle H_{\mathrm{Box}} \psi_{\mathrm{I}} | \psi_{\mathrm{II}} \rangle = \langle \psi_{\mathrm{II}} | H_{\mathrm{Box}} \psi_{\mathrm{I}} \rangle^*, \tag{6}$$

when ψ_{I} and ψ_{II} are from the domain of H_{Box} (here $(*)$ refers to complex conjugation). The Hermitian property of γ^0 and a simple integration by parts give

$$\left(\psi_{\mathrm{II}}^\dagger \gamma^0 \psi_{\mathrm{I}}\right)^* = \psi_{\mathrm{I}}^\dagger \gamma^0 \psi_{\mathrm{II}} \tag{7}$$

and

$$\int_0^L \mathrm{d}z \, \frac{\partial \psi_{\mathrm{I}}^\dagger}{\partial z} \left[\mathrm{i}\gamma^0\gamma^3\right] \psi_{\mathrm{II}} = \psi_{\mathrm{I}}^\dagger \left[\mathrm{i}\gamma^0\gamma^3\right] \psi_{\mathrm{II}} \Big|_0^L - \int_0^L \mathrm{d}z \, \psi_{\mathrm{I}}^\dagger \mathrm{i}\gamma^0\gamma^3 \frac{\partial \psi_{\mathrm{II}}}{\partial z}. \tag{8}$$

Here $\psi^\dagger$ is the adjoint of the column vector (as opposed to the Dirac adjoint $\bar{\psi} = \psi^\dagger \gamma^0$). Eqs. (7) and (8) imply that for (6) to be satisfied, we need the eigenfunctions to satisfy

$$\psi_{\mathrm{I}}^\dagger \left[\mathrm{i}\gamma^0\gamma^3\right] \psi_{\mathrm{II}} \Big|_0^L = 0. \tag{9}$$

As we shall see, the above condition facilitates a self-adjoint Zeno Hamiltonian. Although our eigenfunctions will satisfy this requirement without any added effort, care is taken in pointing this condition for the Hermitian property of the Hamiltonian, because this is precisely the condition that will make our calculations of quantum Zeno dynamics possible.

In this section, we shall point out the essential details in computing $\psi_{+,j}$ (the calculations for the other states are essentially a reproduction with the appropriate free particle Dirac spinors in (5)). The probability flux of the wave function inside the box is given by

$$J_z = \bar{\psi}_{+,j} \gamma^3 \psi_{+,j} = \frac{2k}{E+m} \left[|A_{+,j}|^2 - |B_{+,j}|^2\right]. \tag{10}$$

Since the flux vanishes outside the box, the constancy of probability flux implies that $J_z = 0$. This can be achieved by requiring that $B_{+,j} = e^{\mathrm{i}b_{+,j}} A_{+,j}$. Just as $A_{+,j}$, and $B_{+,j}$, the constant $b_{+,j}$ in the exponential is a function of the label k. The only two undetermined coefficients left are $b_{+,j}$ and $A_{+,j}$. Unlike the usual treatments of the "particle in a box" problem, we do not impose boundary conditions on the wavefunctions directly (for example, see [4] for the relativistic case, and [3] for the non-relativistic case). Instead we impose the condition that

$$e^{\mathrm{i}b_{+,j}(k)} = -1 \quad \text{for all } k. \tag{11}$$

The above condition might seem arbitrary at the moment, but in the end we will see that the eigenfunctions we obtain are dense in the relevant Hilbert space. Using (11) it is easy to show that states with different energy labels are orthogonal provided the only values taken on by the variable k are given by:

$$k = k_n = \frac{n\pi}{L}, \tag{12}$$

where n takes on any integer values. Consequently,

$$E = E_n = \sqrt{k_n^2 + m^2}. \tag{13}$$

Upon normalization, and along with a suitable choice of phase factor, the wavefunction becomes

$$|\psi_{+,j,n}\rangle = \sqrt{\frac{E_n + m}{LE_n}} \begin{pmatrix} \sin(k_n z)\chi_j \\ \frac{-\mathrm{i}k_n}{E_n+m}\cos(k_n z)\chi_j \end{pmatrix} \tilde{\chi}_A. \tag{14}$$

The negative energy states are obtained by the same means. The only exception is that we use a different relative phase between the corresponding spinors in (5), namely,

$$e^{\mathrm{i}b_{-,j}(k)} = 1 \quad \text{for all } k. \tag{15}$$

The resulting wavefunctions are

$$|\psi_{-,j,n}\rangle = \sqrt{\frac{E_n + m}{LE_n}} \begin{pmatrix} \frac{-k_n}{E_n+m}\sin(k_n z)\chi_j \\ \mathrm{i}\cos(k_n z)\chi_j \end{pmatrix} \tilde{\chi}_A. \tag{16}$$

Here $\psi_{-,j,n}(z,t) = \psi_{-,j,n}(z)e^{\mathrm{i}E_n t}$ since $H_{\mathrm{Box}}\psi_{-,j,n} = -E_n\psi_{-,j,n}$ $(j = 1,2)$ because of (13). Eqs. (14) and (16) give a complete list of eigenfunctions for a Dirac particle in a box.

In order to appreciate the Hilbert space formed by the states of a relativistic particle in a box, we define the following spinors:

$$|\psi_{\mathrm{up},j,n}\rangle = \sqrt{\frac{E_n+m}{2E_n}}\left[|\psi_{+,j,n}\rangle + \frac{k_n}{E_n+m}|\psi_{-,j,n}\rangle\right] = \sqrt{\frac{2}{L}}\begin{pmatrix} \sin(k_n z)\chi_j \\ 0 \\ 0 \end{pmatrix}\tilde{\chi}_A, \tag{17}$$

$$|\psi_{\mathrm{down},j,n}\rangle = \mathrm{i}\sqrt{\frac{E_n+m}{2E_n}}\left[\frac{k_n}{E_n+m}|\psi_{+,j,n}\rangle - |\psi_{-,j,n}\rangle\right] = \sqrt{\frac{2}{L}}\begin{pmatrix} 0 \\ 0 \\ \cos(k_n z)\chi_j \end{pmatrix}\tilde{\chi}_A. \tag{18}$$

Since $\{\sin(k_n z) \mid k_n = n\pi/L,\ n \in \mathbb{N}\}$, and $\{\cos(k_n z) \mid k_n = n\pi/L,\ n \in \mathbb{N}\}$ form a basis for $L^2(A)$, from (17) and (18) we find that the closed linear span of (14) and (16), and hence the Hilbert Space for our particle in a box is given by

$$\mathfrak{H}_{\mathrm{Box}} = \left\{ \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \middle|\ \psi_j \in L^2(\mathbb{R}) \text{ and } \psi_j \overset{\text{a.e.}}{\equiv} 0 \text{ in } A^c = \mathbb{R}\setminus A,\ j = \overline{1,4} \right\}, \tag{19}$$

along with the inner product

$$\langle\phi|\psi\rangle = \sum_{j=1}^{4}\int_{-\infty}^{\infty} \mathrm{d}z\, \phi_j^*\psi_j = \sum_{j=1}^{4}\int_{0}^{L} \mathrm{d}z\, \phi_j^*\psi_j. \tag{20}$$

This justifies our assumptions (11), and (15), since we can now expand any function in $\mathfrak{H}_{\mathrm{Box}}$ using our bases (as long as the wavefunction vanishes outside the box). The domain of our Hamiltonian is taken to be

$$\mathrm{Dom}(H_{\mathrm{Box}}) = \left\{\psi \in \mathfrak{H}_{\mathrm{Box}} \mid \psi_j \text{ is a.c. in } A,\ j = \overline{1,4},\ \psi_j(0) = \psi_j(L) = 0,\ j = 1,2\right\}. \tag{21}$$

Since the eigenfunctions (14) and (16) belong to the $\mathrm{Dom}(H_{\mathrm{Box}})$, it is clear that the operator H_{Box} is densely defined and thus symmetric. If $\varphi \in \mathrm{Dom}(H^{\dagger}_{\mathrm{Box}})$, then (9) would imply that

$$\varphi_1(L)\psi_3(L) - \varphi_2(L)\psi_4(L) - \varphi_1(0)\psi_3(0) + \varphi_2(0)\psi_4(0) = 0, \tag{22}$$

for every $\psi \in \mathrm{Dom}(H_{\mathrm{Box}})$. This can happen if and only if the first two components of $\varphi \in \mathrm{Dom}(H^{\dagger}_{\mathrm{Box}})$ vanish at the end points. These are precisely all the spinors contained in the domain of H_{Box}, and hence, $\mathrm{Dom}(H_{\mathrm{Box}}) = \mathrm{Dom}(H^{\dagger}_{\mathrm{Box}})$. Thus H_{Box}, as we have defined it, is a self-adjoint operator. It is also clear that, conditions (11) and (15) impose Dirichlet boundary conditions on the large components of the wavefunctions.

The non-relativistic limit of the positive energy states are manifest: here $E_n \approx m$, and $\frac{k_n}{E_n+m} \approx 0$. Therefore, the large components of the spinors (14) become

$$|\psi_{+,j,n}\rangle \xrightarrow{\mathrm{NR}} |\psi_{\mathrm{NR},j,n}\rangle = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}z\right)\chi_j\tilde{\chi}_A, \tag{23}$$

which is precisely the form given in [3] for the non relativistic particle in a box. Here, the subscript NR in the wavefunction refers to the non-relativistic limit of the large components of the positive energy Dirac spinors of the particle in a box. We conclude our analysis of the particle in a box by reiterating that our wavefunctions (14), and (16) does indeed satisfy the condition specified by (9).

3. Relativistic quantum Zeno dynamics

The notion of "Zeno's paradox" as it was called was initially put forth by Misra and Sudarshan (MS theorem) [2]. The purpose of this analysis is to test the time evolution of a quantum system under constant observation. Recently, Facchi et al. [2] has been able to discuss the QZD of a Schrödinger type system with relative ease. This relied on the use of a preferred bases set of the Hilbert space, namely: the eigenfunctions of the Zeno Hamiltonian. The Zeno Hamiltonian, as it turns out is nothing more than the Schrödinger Hamiltonian for the particle in a box.

In this section, we wish to demonstrate the QZD of the Dirac Hamiltonian. While this is a specific example, it does extend the theory put forth by Misra and Sudarshan. The MS theorem is proven in general for a Hamiltonian that is semi-bounded. The Dirac operator is not. The problem regarding a not lower semi-bounded operator was considered in some detail by Facchi, Gorini et al. [7]. They, however, were not concerned with the Dirac operator. We show that the Dirac operator, although unbounded, exhibits QZD.

The tools we need to compute the QZD of the Dirac Hamiltonian is essentially contained in the paper by Facchi [2]. As mentioned in the first paragraph of this section, the availability of a preferred bases of the Zeno Hamiltonian makes the calculations reasonable. In our case, the Zeno Hamiltonian is nothing more than the "Dirac particle in a box" operator. This was exactly what was developed in the previous section.

Since the topic of QZD has been well treated in earlier publications [1,2,7], we will restrict the introductory remarks to just serve our purposes. Consider a wavefunction ψ_0 initially contained in the box of the previous section. Let G_0 denote the propagator for the Dirac operator (1). The wavefunction at some later time t is given by

$$\psi(z',t) = \int G_0(z',t;z)\psi_0(z)\,\mathrm{d}z = \int G_0(z',t;z)\tilde{\chi}_A(z)\psi_0(z)\,\mathrm{d}z, \tag{24}$$

where $\tilde{\chi}_A(z)$ is the characteristic function defined in (4). If we now consider the effect of projecting the wavefunction onto our box, the resulting state is just given by the natural projection:

$$\psi(z',t) \to \psi(z',t)\tilde{\chi}_A(z'). \tag{25}$$

The new state can be thought of as being generated with the modified propagator:

$$G_0(z',t;z) \to \tilde{\chi}_A(z')G_0(z',t;z)\tilde{\chi}_A(z) = G(z',t;z). \tag{26}$$

We now define what is meant by continuous observation of a quantum system. Lets divide the time interval T into N equal parts, and set $t = T/N$. The wavefunction after N repeated projections of the previous type at set time interval t is given by

$$\psi(z', T) = \int \mathrm{d}y_{N-1} \cdots \int \mathrm{d}y_1 \int \mathrm{d}z\, G(z', t; y_{N-1}) \cdots G(y_2, t; y_1) G(y_1, t; z) \psi_0(z). \tag{27}$$

By continuous observation, we mean the $N \to \infty$ limit of (27). In general, it is not clear whether such limit even exists. For the case of the Dirac operator we will show that the limit does indeed exists, and we will calculate the resulting propagator. In (27), we may as well dispense with the information of the initial wavefunction, and focus instead on the kernel

$$K(z', T; z) = \lim_{N\to\infty} \int \mathrm{d}y_{N-1} \cdots \int \mathrm{d}y_1\, G(z', t; y_{N-1}) \cdots G(y_1, t; z). \tag{28}$$

If the Hamiltonian governing the quantum mechanical system is such that the above limit exists, and if the resulting Kernel is given by a unitary operator, we say that the system exhibits QZD. We begin our analysis by calculating the propagator G_0. Here, G_0 is the propagator for the free Dirac operator H_{D}. Let $|\psi_{\pm,j}(k)\rangle$ for $j = 1, 2$ represent the usual (for example, see [6]) orthonormal set of eigenfunctions of H_{D} with one spatial dimension. Here, $\pm$ subscript refers to the positive and negative eigenstates, respectively, $j = 1, 2$ refers to the spin up and spin down states as well. The parameter k can take on any real values, and they correspond the momentum eigenvalue. It is not difficult to see that:

$$\sum_j \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \Big[\big|\psi_{+,j}(k)\big\rangle\big\langle\psi_{+,j}(k)\big| + \big|\psi_{-,j}(-k)\big\rangle\big\langle\psi_{-,j}(-k)\big| \Big] = 1. \tag{29}$$

For clearity,

$$\begin{aligned}
&\big\langle z'\big| \sum_j \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \Big[\big|\psi_{+,j}(k)\big\rangle\big\langle\psi_{+,j}(k)\big| + \big|\psi_{-,j}(-k)\big\rangle\big\langle\psi_{-,j}(-k)\big| \Big] |z\rangle \\
&\quad = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \frac{1}{2E_k} \big[\big(\gamma^0 E_k - \gamma^3 k + m\big) - \big(-\gamma^0 E_k - \gamma^3 k + m\big) \big] \gamma^0 e^{\mathrm{i}k(z'-z)} \\
&\quad = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \frac{1}{2E_k} \big(2\gamma^0 E_k\big) \gamma^0 e^{\mathrm{i}k(z'-z)} = \delta(z'-z) I_4.
\end{aligned}$$

Here $E_k = \sqrt{k^2 + m^2}$ and I_4 is the (4×4) identity matrix. The propagator G_0 is given by $G_0(x, t; y) = \langle x|U(t)|y\rangle$, where U is the unitary operator generated by H_{D}, i.e., $U(t) = \exp(-\mathrm{i}H_{\mathrm{D}}t)$. Using (29), we find that

$$\begin{aligned}
G_0(x, t; y) &= \langle x| \sum_j \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \Big[\big|\psi_{+,j}(k)\big\rangle\big\langle\psi_{+,j}(k)\big| + \big|\psi_{-,j}(-k)\big\rangle\big\langle\psi_{-,j}(-k)\big| \Big] U(t)|y\rangle \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \frac{1}{2E_k} \big[\big(\gamma^0 E_k - \gamma^3 k + m\big) e^{-\mathrm{i}E_k t} - \big(-\gamma^0 E_k - \gamma^3 k + m\big) e^{\mathrm{i}E_k t} \big] \gamma^0 e^{\mathrm{i}k(x-y)}.
\end{aligned} \tag{30}$$

In the above equation, the time dependence of the propagator is calculated by noting that

$$\langle\psi_{\pm,j}(k)|U(t)|y\rangle = \exp\big(-\mathrm{i}(\pm E_k t)\big) \langle\psi_{\pm,j}(k)|y\rangle$$

since $H_D|\psi_{\pm,j}(k)\rangle = \pm E_k|\psi_{\pm,j}(k)\rangle$. The propagator G is given by (26). Simplifying (30), we find that

$$G(x,t;y) = \tilde{\chi}_A(x)\tilde{\chi}_A(y)\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{1}{2E_k}\left[2\gamma^0 E_k\cos(E_k t) + 2i\left(\gamma^3 k - m\right)\sin(E_k t)\right]\gamma^0 e^{ik(x-y)}. \tag{31}$$

From (28), we see that there are N many such propagators in the Kernel, and since we are only interested in the $N\to\infty$ limit, and since $t = T/N$, we need only to approximate the above propagator to first order in t. Therefore,

$$\begin{aligned} G(x,t;y) &\approx \tilde{\chi}_A(x)\tilde{\chi}_A(y)\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{1}{2E_k}\left[2\gamma^0 E_k + 2i\left(\gamma^3 k - m\right)E_k t\right]\gamma^0 e^{ik(x-y)} + O\left(t^2\right) \\ &= \tilde{\chi}_A(x)\tilde{\chi}_A(y)\left[\delta(x-y) + it\int_{-\infty}^{\infty}\frac{dk}{2\pi}\left(\gamma^3 k - m\right)\gamma^0 e^{ik(x-y)}\right] + O\left(t^2\right). \end{aligned} \tag{32}$$

The above propagator has a compact support in the interval $[0, L]$. Therefore, all relevant information is obtained by calculating its matrix elements between a set of spinors dense in $L^2(A)$. From (21) it clear that the spinors (14) and (16) are exactly what we need, and so we set out to evaluate elements of the type

$$\langle\psi_{\pm,k,m}|G|\psi_{\pm,j,n}\rangle = \int dx\, dy\,\langle\psi_{\pm,k,m}|x\rangle\langle x|G|y\rangle\langle y|\psi_{\pm,j,n}\rangle. \tag{33}$$

For definiteness, let us consider $\langle\psi_{+,k,m}|G|\psi_{+,j,n}\rangle$. The remaining elements can be evaluated in a like manner (of which, we will just give the results). The zeroth order term yields identity because of the delta function. In order to simplify the first order (in t) term in (32) we note that

$$m\gamma^0\langle y|\psi_{+,j,n}\rangle = \left[E_n + i\gamma^0\gamma^3\frac{d}{dy}\right]\langle y|\psi_{+,j,n}\rangle, \tag{34}$$

$$\begin{aligned} &\int_0^L \frac{dk'}{2\pi}\langle\psi_{+,k,m}|x\rangle\left[i\gamma^0\gamma^3\frac{d}{dy}\right]\langle y|\psi_{+,j,n}\rangle e^{ik'(x-y)}\,dx\,dy \\ &\quad= \int_0^L \frac{dk'}{2\pi}\langle\psi_{+,k,m}|x\rangle\left[i\gamma^0\gamma^3\right]\langle y|\psi_{+,j,n}\rangle e^{ik'(x-y)}\,dx\Big|_{y=0}^{y=L} \\ &\qquad- \int_0^L \frac{dk'}{2\pi}\langle\psi_{+,k,m}|x\rangle\left[i\gamma^0\gamma^3\right]\langle y|\psi_{+,j,n}\rangle\frac{d}{dy}e^{ik'(x-y)}\,dx\,dy \\ &\quad= \delta(x-y)\langle\psi_{+,k,m}|y\rangle\left[i\gamma^0\gamma^3\right]\langle y|\psi_{+,j,n}\rangle\Big|_{y=0}^{y=L} + \int_0^L\frac{dk'}{2\pi}\langle\psi_{+,k,m}|x\rangle\left[-\gamma^0\gamma^3 k'\right]\langle y|\psi_{+,j,n}\rangle e^{ik'(x-y)}\,dx\,dy, \end{aligned} \tag{35}$$

which follows from a simple integration by parts. The first term in the right-hand side of (35) vanishes due to the Hermitian property (9). Therefore,

$$\int_0^L \frac{dk'}{2\pi}\langle\psi_{+,k,m}|x\rangle\left[i\gamma^0\gamma^3\frac{d}{dy}\right]\langle y|\psi_{+,j,n}\rangle e^{ik'(x-y)}\,dx\,dy$$
$$=\int_0^L \frac{dk'}{2\pi}\langle\psi_{+,k,m}|x\rangle\left[-\gamma^0\gamma^3k'\right]\langle y|\psi_{+,j,n}\rangle e^{ik'(x-y)}\,dx\,dy. \tag{36}$$

Substituting (32), (34), and (36) in (33) we finally get that

$$\langle\psi_{+,k,m}|G|\psi_{+,j,n}\rangle=(1-iE_nt)\delta_{mn}\delta_{kj}+O(t^2). \tag{37}$$

Similarly,

$$\langle\psi_{-,k,m}|G|\psi_{-,j,n}\rangle=\left[1-i(-E_n)t\right]\delta_{mn}\delta_{kj}+O(t^2). \tag{38}$$

Here, it is important to remember that $-E_n$ is the energy of the negative energy states. Finally,

$$\langle\psi_{-,k,m}|G|\psi_{+,j,n}\rangle=\langle\psi_{+,k,m}|G|\psi_{-,j,n}\rangle=0+O(t^2). \tag{39}$$

The time evolution of quantum states under continuous observation for a finite time T is given by the Kernel

$$\langle\psi_{\pm,k,m}|K|\psi_{\pm,j,n}\rangle=\int dz'\,dz\,\langle\psi_{\pm,k,m}|z'\rangle K(z',T;z)\langle z|\psi_{\pm,j,n}\rangle.$$

Using (37)–(39), we find that

$$\langle\psi_{+,k,m}|K|\psi_{+,j,n}\rangle=e^{-iE_nT}\delta_{mn}\delta_{kj},\qquad \langle\psi_{-,k,m}|K|\psi_{-,j,n}\rangle=e^{-i(-E_n)T}\delta_{mn}\delta_{kj},$$
$$\langle\psi_{+,k,m}|K|\psi_{-,j,n}\rangle=\langle\psi_{-,k,m}|K|\psi_{+,j,n}\rangle=0. \tag{40}$$

The above Kernel is precisely the propagator for the self-adjoint Hamiltonian H_{Box}, thus yielding a unitary dynamics under continuous observation. This is the main result of the section. The relativistic "particle" evolving under the Dirac Hamiltonian under constant observation within a box does not ever leave the box. That is not to say that the time evolution is trivial. The particle behaves as if it is subject to an external potential of the type $\gamma^0M[1-\tilde{\chi}_A(x)]$ in the infinite M limit. This would follow immediately from the MS theorem were it not for that fact the Dirac operator is not lower semi-bounded. Since this is only a specific example, perhaps the direction in which the MS theorem can be extended is by relaxing the requirement that the Hamiltonian operator has to be lower semi-bounded.

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